First order theories

A *first order theory* of a language L is just a set T of L-formulas. Sometimes we call formulas in T the *axioms*.

Examples:

Group theory The language of group theory has one constant e and one binary function symbol $\cdot$. The theory of group theory has axioms

$$\forall x \exists y (x \cdot y = e \wedge y \cdot x = e)$$
$$\forall x \forall y \forall z \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z .$$
$$\forall x \left(x \cdot e = x \wedge e \cdot x = x\right)$$

The theory of *abelian groups* moreover has the following axiom

$$\forall x \forall y \quad x \cdot y = y \cdot x .$$

Here and in what follows we shall use variables $x, y, z, \ldots$ instead of $x_0, x_1, x_2, \ldots$ in order to simplify notation.

The theory of <u>linearly</u> <u>ordered</u> <u>sets</u> is a theory of the language with a single binary relation symbol $<$ and with axioms

$$\forall x \ \neg(x<x)$$
$$\forall x \forall y \forall z \ ((x<y \wedge y<z) \rightarrow x<z)$$
$$\forall x \forall y \ (x=y \vee x<y \vee y<x).$$

The expressive power of first order logic is greater than that of propositional logic.

<u>Example</u> Formalise the following in first order logic:

"If every ancestor of an ancestor of an individual is also an ancestor of the same individual and no individual is his/her own ancestor, then there is someone without any ancestors."

Here $L = \{R\}$, where R is a binary relation symbol:

$Rxy \iff x$ is an ancestor of y.

Formalised we have:

$$\Big(\Big[\Big(\forall x\, \forall y\, \forall z\, \big((Rxy \wedge Ryz) \to Rxz\big)\Big) \wedge \forall x\, \neg Rxx\Big] \to \exists x\, \forall y\, \neg Ryx\Big)$$

<u>Example</u>